Determination of the Survival Probability of a Launch Vehicle Rising Through a Random Wind Field

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1. Introduction

An important aspect of the structural design of a launch vehicle is the determination of the probability that the vehicle will survive its flight through the atmosphere. If for each component of the structure the critical load which will cause the component to fail has been determined, then the probability of survival of the vehicle is the probability that none of these critical loads will be exceeded during its flight.

For final design purposes the current practice is to determine the probability of failure of a given structural component from a "statistical load survey".[1] A large number of representative wind profiles are assembled, and for each of them the corresponding time history of the load applied to the given component is computed. If the critical load is exceeded in n% of the records thus obtained, the probability of failure of the component is estimated to be n/100 and the critical load is said to be an n% load.

While this method is straightforward, it makes only a very limited use of the statistical information available from the load survey. Since the probability of survival of a given structural component has to be larger than the probability of survival of the entire vehicle, one should expect to have to consider in many cases loads of 1% or less. This means that, in order to obtain a reliable estimate of the probability of failure of a component, one will have to determine the time history of the load for several hundred representative wind profiles.

In the first part of this paper we shall present a method which makes a more effective use of the information contained in the statistical load surveys. As a result, fewer records will be required to determine the probability of failure of a given structural component. If certain requirements are met, it will even be possible to determine this probability from groups of records in which the critical load was never exceeded.

In the second part of the paper we shall show how the use of a statistical load survey may be avoided altogether when the launch vehicle may be assumed to constitute a linear system. The probability of survival of the vehicle may then be determined directly from the statistical properties of the wind profiles.

I. Determination of the Survival Probability of the Vehicle from Certain Statistical Parameters of its Response.

2. Formulation of the Problem

Let $q(z)$ be the load applied to a structural component of the vehicle as the vehicle reaches the height z during a given flight. Assuming that failure of the component will occur if a critical value of the load is reached during the flight, we note that the probability of survival of the component is the probability that the nonstationary random function $q(z)$ will not cross the level $q = \alpha$ over the interval $0 < z < z_m$, where z_m is the maximum height reached by the vehicle. Assuming no interaction between the various components of the vehicle, the probability of survival of the vehicle itself may then be obtained by forming the product of the survival probabilities of its components. We should note that the above approach to the problem may be extended to include failure criteria other than critical "loads". These criteria could involve structural parameters such as bending moments, stresses, or displacements. Therefore, in the following discussion the function $q(z)$ will represent the variation during the vehicle's flight of some significant but unspecified structural parameter and will simply be referred to as the "response" of the vehicle.

Let us denote by $P_\alpha(z)dz$ the probability computed over an ensemble of records that the response $q(z)$ will exceed the level $q = \alpha$ for the first time between the elevations z and $z + dz$. The sum of these probabilities as z varies from 0 to z_m yields the probability of failure of the component under consideration ; the probability of survival S_α of the component may then be expressed as

$$S_\alpha = 1 - \int_0^{z_m} P_\alpha(z)\, dz \qquad (1)$$

Thus the problem of determining the survival probability S_α reduces to finding an expression for the probability density $P_\alpha(z)$ of a first occurence of the value α of the vehicle response at height z.

3. Approximate Expression for $P_\alpha(z)$

The difficulty of obtaining a tractable expression for $P_\alpha(z)$ becomes apparent if we note that for $\alpha = 0$ the problem reduces to the classical zero-crossing problem for which no general solution is known to exist.[2] We shall therefore derive an approximate expression for $P_\alpha(z)$ which should be valid for sufficiently large values of α.

Following an approach similar to that used by Rice and Beer in the case of a stationary process[3], we shall express $P_\alpha(z)$ as the product

$$P_\alpha(z) = f_\alpha\ (z|\ 0<r<z)\ [1 - \int_0^z P_\alpha(r)\, dr] \qquad (2)$$

where $f_\alpha(z \mid 0<r<z)\,dz$ is the probability of a crossing of $q = \alpha$ in the interval z, $z + dz$, given no prior crossing, and where the expression between brackets represents the probability of no crossing for $0<r<z$. Assuming the initial condition $q = 0$ (for $z = 0$), and thus $P_\alpha(0) = 0$, and solving the integral equation (2), we obtain

$$P_\alpha(z) = f_\alpha(z \mid 0<r<z) \exp\left[-\int_0^z f\,(s \mid 0<r<s)\,ds\right] \qquad (3)$$

which is an exact expression for the probability density $P_\alpha(z)$ of a first crossing of $q = \alpha$ at z.

In order to obtain a more tractable expression for $P_\alpha(z)$, we replace the function $f_\alpha(z \mid 0<r<z)$ by $p_\alpha(z)$, where $p_\alpha(z)\,dz$ is the expected number of upward crossings of $q = \alpha$ in the interval z, $z + dz$, and obtain the approximation

$$P_\alpha(z) = p_\alpha(z) \exp\left[-\int_0^z p_\alpha(r)\,dr\right] \qquad (4)$$

The assumptions implied by this approximation will be indicated in the Appendix. Let us note here, however, that substituting $p_\alpha(z)$ for $f_\alpha(z \mid 0<r<z)$ amounts to neglecting the condition of no crossing of $q = \alpha$ over the interval 0, z in the first factor of the right-hand member of Eq. (2).

It is well known [4] that the expected number of α-crossings per unit time of a stationary random function of time may be expressed in terms of the joint probability density of the function and its derivative. In a similar way, we shall express the function $p_\alpha(z)$ as

$$p_\alpha(z) = \int_0^\infty \beta g\ (\alpha, \beta;\ z)\ d\beta \tag{5}$$

The function $g(\alpha, \beta;\ z)$ is the z-dependent joint density defined by

$$g(\alpha, \beta;\ z)\ d\alpha d\beta = \text{Probability that } \begin{array}{l} \alpha < q(z) < \alpha + d\alpha \ \text{ and} \\ \beta < \dot{q}(z) < \beta + d\beta \end{array} \tag{6}$$

where $\dot{q}(z)$ is the derivative of the response $q(z)$ with respect to the height z.

4. Approximate Expression for the Survival Probability S_α

Substituting into (1) the approximate expression (4) obtained for $P_\alpha(z)$, we write

$$S_\alpha = 1 - \int_0^{zm} p_\alpha(z)\ \exp\ [-\int_0^z p_\alpha(r)\ dr]\ dz$$

Noting that

$$p_\alpha(z)\ \exp\ [-\int_0^z p_\alpha(r)\ dr] = -\frac{d}{dz}\ \exp\ [-\int_0^z p_\alpha(r)\ dr]$$

we obtain

$$S_\alpha = \exp \left[-\int_0^{zm} p_\alpha(z)\, dz\right] \tag{7}$$

where the integral represents the expected total number of crossings of $q = \alpha$ during the flight of the vehicle. Substituting from (5) for $p_\alpha(z)$, we may also write

$$S_\alpha = \exp \left[-\int_0^{zm} \int_0^\infty \beta g(\alpha, \beta;\ z)\, d\beta dz\right] \tag{8}$$

where $g(\alpha, \beta;\ z)$ is the z-dependent joint probability density defined in (6).

The expressions (7) and (8) obtained for the survival probability S_α of a given component of the vehicle are approximate. They will be valid under the same condition as the approximation (4), namely the condition that the value $q = \alpha$ at which failure occurs be sufficiently high, as discussed in the Appendix. It should also be noted that the method just presented is applicable only if it is possible to determine from the available records the joint probability density $g(\alpha, \beta;\ z)$ of the response $q(z)$ and its derivative for each height z.

5. Application to the Case of a Normal Distribution of the Response and its Derivative

If $q(z)$ and $\dot{q}(z)$ are normally distributed random variables with means $\langle q(z) \rangle$ and $\langle \dot{q}(z) \rangle$, where $\langle\ \rangle$ represents an average taken over an ensemble of flights, then

$$g(\alpha,\beta;\ z) = \frac{1}{2\pi|M|^{1/2}} \exp\left[- \frac{1}{2|M|} (d_{22}r^2 + 2d_{12}rs + d_{11}s^2)\right] \quad (9)$$

where

$$r = \alpha - \langle q(z) \rangle , \qquad s = \beta - \langle \dot{q}(z) \rangle$$

and where the d_{ij} represent the variances and covariance

$$\begin{aligned} d_{11}(z) &= \langle [q(z) - \langle q(z) \rangle]^2 \rangle \\ d_{12}(z) &= \langle [q(z) - \langle q(z) \rangle] [\dot{q}(z) - \langle \dot{q}(z) \rangle] \rangle \\ d_{22}(z) &= \langle [\dot{q}(z) - \langle \dot{q}(z) \rangle]^2 \rangle \end{aligned} \quad (10)$$

while $|M|$ is the determinant

$$|M| = \begin{vmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{vmatrix} \quad (11)$$

We may note the following relation between the variance $d_{11}(z)$ and the covariance $d_{12}(z)$:

$$d_{12}(z) = \frac{1}{2} \frac{d}{dz} d_{11}(z) \quad (12)$$

Introducing the covariance function

$$\mathcal{C}_{qq}(z_1, z_2) = \langle[q(z_1) - \langle q(z_1)\rangle]\ [q(z_2) - \langle q(z_2)\rangle]\rangle \qquad (13)$$

we also note that the d_{ij} may be expressed in terms of the covariance function and its partial derivatives:

$$d_{11}(z) = \mathcal{C}_{qq}(z, z)$$

$$d_{12}(z) = \left.\frac{\partial \mathcal{C}_{qq}(z_1, z_2)}{\partial z_1}\right|_{z_1 = z_2 = z} \qquad (14)$$

$$d_{22}(z) = \left.\frac{\partial^2 \mathcal{C}_{qq}(z_1, z_2)}{\partial z_1 \partial z_2}\right|_{z_1 = z_2 = z}$$

Substituting for $g(\alpha, \beta;\ z)$ from (9) into (5), we obtain after integration

$$p_\alpha(z) = \frac{|M|^{1/2}}{2\pi d_{11}} [e^{-\rho^2} + \sqrt{\pi}\rho(1 + \operatorname{erf}\ \rho)]\ e^{\frac{-[\alpha - \langle q(z)\rangle]^2}{2d_{11}}} \qquad (15)$$

where

$$\rho = \left[\frac{d_{12}}{d_{11}}[\alpha - \langle q(z)\rangle] + \langle\dot{q}(z)\rangle\right]\left[\frac{d_{11}}{2|M|}\right]^{1/2} \qquad (16)$$

We note that, if $q(z)$ is stationary,

$$\langle q(z)\rangle = c,\quad \langle\dot{q}(z)\rangle = 0,\quad d_{12} = 0$$

and Eq. (15) reduces to

$$p_\alpha = \frac{1}{2\pi}\sqrt{\frac{d_{22}}{d_{11}}}\, e^{\frac{-(\alpha-c)^2}{2d_{11}}} \tag{17}$$

which is the expression derived by S.O. Rice[4] for the expected number of upward α-crossings per unit height.

Substituting now for $p_\alpha(z)$ from (15) into (7) we obtain the following expression for the probability of survival in the case of a normal distribution:

$$S_\alpha = \exp\left[-\int_0^{z_m} \frac{|M|^{1/2}}{2\pi d_{11}} [e^{-\rho} + \sqrt{\pi}\rho(1+\mathrm{erf}\ \rho)]\, e^{\frac{-[\alpha-\langle q(z)\rangle]^2}{2d_{11}}} dz\right] \tag{18}$$

where d_{11}, the determinant $|M|$, and ρ are functions of z defined respectively in (10), (11), and (16).

II. Determination of the Survival Probability of the Vehicle from its Dynamic Characteristics and from the Statistical Characteristics of the Wind Field.

6. Statement of Assumptions and Outline of the Proposed Methods

We shall now consider the case when the structure of the launch vehicle may be approximated by a linear model, and when the wind-velocity $u(z)$ may be considered a normally distributed random variable. Since the response of a linear system to a Gaussian input is itself Gaussian, the response $q(z)$ of the vehicle and its derivative $\dot{q}(z)$ will be normally distributed and Eq. (18) will apply.

As seen in the preceding section, all parameters contained in Eq. (18) may be expressed in terms of the mean value $\langle q(z) \rangle$ of the response and of the covariance function $\varrho_{qq}(z_1, z_2)$ and its derivatives. Since the system is assumed linear, the mean value $\langle q(z) \rangle$ may be obtained by computing the response of the system to the mean value $\langle u(z) \rangle$ of the wind velocity at each elevation, and the covariance function $\varrho_{qq}(z_1, z_2)$ may be expressed as the double convolution of the covariance function $\varrho_{uu}(z_1, z_2)$ of the wind-velocity field and of the impulse-response function of the system. This will be described in detail in Sections 7 and 8.

An alternate method will be presented in Section 9. It is based on the determination of the generalized power spectrum of the wind-velocity field and on the use of a frequency-response function which depends upon the elevation of the vehicle.

It should be noted that the proposed methods eliminate the need for two separate analyses to predict the effects of wind shear and of atmospheric gustiness on the launch vehicle. If a sufficiently refined wind-velocity measurement technique - such as the smoke-trail method - is used in compiling the statistics of the wind field, the effects of both wind shear and atmospheric gustiness will be taken into account in the computation of the survival probability of the vehicle.

Moreover, the proposed methods have the advantage of providing for the separate determination of the statistical characteristics of the atmosphere and of the dynamic characteristics of the vehicle before their combined effect on the response of the vehicle is estimated. Thus wind-velocity statistics compiled at a given location may be used to determine the survival probability of several vehicles, while the dynamic analysis carried on a given vehicle will be used to predict its response under various types of atmospheric conditions.

7. Impulse-Response Function Method

As indicated above, we shall assume that the response $q(z)$ of the vehicle may be defined as the solution of a linear differential equation

$$L\, q(z) = u(z) \tag{19}$$

where L represents a differential operator with z-dependent coefficients and $u(z)$ the wind velocity profile. The response $q(z)$ may be expressed as the convolution

$$q(z) = \int_0^z h(z', z)\, u(z')\, dz' \tag{20}$$

of the wind profile and of the function $h(z', z)$ representing the solution of the differential equation

$$L\, q(z) = \delta(z - z') \tag{21}$$

The function $\delta(z - z')$ is the Dirac delta function and represents a unit wind impulse at height z'. The corresponding response $h(z', z)$ will be referred to as the impulse-response function of the vehicle.

The mean value of the response of the vehicle at height z is obtained by averaging both members of Eq. (20) over the ensemble of flights considered. We thus have

$$\langle q(z) \rangle = \int_0^z h(z', z)\, \langle u(z') \rangle\, dz' \tag{22}$$

and note that the mean response $\langle q(z)\rangle$ may be computed as the response of the vehicle to the mean wind-velocity profile $\langle u(z)\rangle$.

To obtain the covariance function of the response q(z) we subtract Eq. (22) from Eq. (20), member by member,

$$q(z) - \langle q(z)\rangle = \int_0^z h(z', z)\,[u(z') - \langle u(z')\rangle]\,dz' \quad (23)$$

and substitute into Eq. (13). We have

$$\varphi_{qq}(z_1, z_2) = \int_0^{z_1}\int_0^{z_2} h(z_1', z_1)\,h(z_2', z_2)\,\varphi_{uu}(z_1', z_2')\,dz_1'dz_2' \quad (24)$$

where φ_{uu} represents the covariance function of the wind-velocity profile:

$$\begin{aligned}\varphi_{uu}(z_1, z_2) &= \langle[u(z_1) - \langle u(z)\rangle]\,[u(z_2) - \langle u(z_2)\rangle]\rangle \\ &= \langle u(z_1)\,u(z_2)\rangle - \langle u(z_1)\rangle\,\langle u(z_2)\rangle\end{aligned} \quad (25)$$

The functions $d_{ij}(z)$ used for the determination of the survival probability S_α are obtained by substituting for φ_{qq} from (24) into the relations (14). We have

$$d_{11}(z) = \int_0^z\int_0^z h(z_1', z)\,h(z_2', z)\,\varphi_{uu}(z_1', z_2')\,dz_1'dz_2' \quad (26)$$

$$d_{12}(z) = \int_0^z\int_0^z h_z(z_1', z)\,h(z_2', z)\,\varphi_{uu}(z_1', z_2')\,dz_1'dz_2' \quad (27)$$

$$d_{22}(z) = \int_0^z\int_0^z h_z(z_1', z)\,h_z(z_2', z)\,\varphi_{uu}(z_1', z_2')\,dz_1'dz_2' \quad (28)$$

where $h_z(z', z)$ denotes the derivative with respect to z of the impulse-response function of the vehicle. Differentiating both members of Eq. (20) with respect to z, we have

$$\dot{q}(z) = \int_0^z h_z(z', z)\, u(z')\, dz' \tag{29}$$

and observe that $h_z(z', z)$ may also be viewed as the impulse-response function of the system whose output is equal to $\dot{q}(z)$.

In actual computations, the integrals in Eqs. (26), (27), and (28) will be replaced by sums. For instance, the variance $d_{11}(z)$ will be expressed as

$$d_{11}(z) \simeq \sum_{r=1}^{n} \sum_{s=1}^{n} h(z_r', z)\, h(z_s', z)\, \mathcal{C}_{uu}(z_r', z_s')\, (\Delta z)^2 \tag{30}$$

where $n = z/\Delta z$, $z_r' = (r - 1)\Delta z$, and $z_s' = (s - 1)\Delta z$. To evaluate the right-hand member of Eq. (30), it will be necessary to determine $h(z_r', z)$ for $r = 1, 2, \ldots, n$, and thus to solve Eq. (21) for n different values of z'. The number n of responses to be calculated may be estimated by noting that the interval Δz should be appreciably smaller than the smallest wave-length to which the vehicle is expected to respond.

8. Use of the Adjoint System

The number of responses to be computed may be considerably reduced through the use of the adjoint system. This alternate approach takes advantage of the fact that the number of values of z for which the functions $d_{ij}(z)$ must be determined to obtain a good estimate of the survival probability S_α is considerably smaller than the average number n of intervals Δz which must be used in the approximation shown in Eq. (30).

Introducing the variable $\zeta = z - z'$ and defining the new function

$$h(z;\ \zeta) = h(z - \zeta, z) \tag{31}$$

we write the variance in the form

$$d_{11}(z) = \int_0^z\int_0^z h(z;\zeta_1)\ h(z;\zeta_2)\,\mathcal{C}_{uu}(z - \zeta_1,\ z - \zeta_2)\,d\zeta_1 d\zeta_2 \tag{32}$$

which may be approximated by

$$d_{11}(z) \simeq \sum_{r=1}^{n}\sum_{s=1}^{n} h(z;\ \zeta_r)\ h(z;\ \zeta_s)\,\mathcal{C}_{uu}(z-\zeta_r,\ z-\zeta_s)\ (\Delta\zeta)^2 \tag{33}$$

where $n = z/\Delta\zeta$, $\zeta_r = r\Delta\zeta$, and $\zeta_s = s\Delta\zeta$. Now, for any given value of z the function $h(z;\ \zeta)$ depends only upon ζ and may be viewed as the solution of the differential equation

$$L_z\, q(\zeta) = \delta(\zeta) \tag{34}$$

which defines the response of a certain system to a unit impulse applied at $\zeta = 0$. This system is the adjoint of the vehicle system for the given height z and its differential operator L_z may be obtained from the operator L of the vehicle system[5]. Solving Eq. (34) for a particular value of the parameter z will yield the response $h(z; \zeta)$ for all values $\zeta = \zeta_r$ and $\zeta = \zeta_s$ required in the computation of $d_{11}(z)$ for the height z. Thus the number of differential equations to be solved is equal to the number of values of z for which the functions $d_{ij}(z)$ should be determined and, as stated earlier, this number is considerably smaller than the number of values of z' for which the original differential equation (21) should be solved. It should be noted that the determination of the response $h(z; \zeta)$ is particularly simple when an analog simulation of the vehicle system has been obtained.

The computation of the functions $d_{12}(z)$ and $d_{22}(z)$ may be similarly simplified through the use of the adjoint of the system characterized by the impulse-response function $h_z(z', z)$, i.e., the system whose output is equal to the derivative $\dot{q}(z)$ of the response of interest.

9. Frequency-Response Function Method

We shall now consider an alternate method for the computation of the functions $d_{ij}(z)$ which involves the determination of a z-dependent frequency-response function. One of the advantages of the method is that the concept of frequency-response function is more familiar to structural engineers than the concept of impulse-response function. Another is that, in general, a smaller number of characteristic responses need to be computed to determine the survival probability of the vehicle (unless the adjoint system is used). A possible drawback of the method is that the computation of the functions $d_{ij}(z)$ is based on the knowledge of the generalized power spectrum of the wind-velocity field, a function which must be obtained through a double Fourier transformation of the covariance function.

The method is based on the computation of the response of the vehicle to sinusoidal wind profiles of various wave-numbers k. Setting

$$u(z) = e^{ikz} \tag{35}$$

in Eq. (19), we obtain the set of differential equations

$$L\, q(z) = e^{ikz} \tag{36}$$

which define the desired responses. Denoting by q(z; k) the solutions of Eq. (36), we define the frequency-response function H(z; k) of the vehicle by setting

$$q(z;\, k) = H(z;\, k)\, e^{ikz} \tag{37}$$

We note that, contrary to the case of a system characterized by a differential operator with constant coefficients, the frequency-response function of the vehicle depends upon the variable z.

Substituting for $q(z)$ and $u(z)$, respectively, from (37) and (35) into (20), and dividing both members by e^{ikz}, we obtain the following relation between the frequency-response function and the impulse-response function

$$H(z;\ k) = \int_0^z h(z',\ z)\ e^{-ik(z-z')}dz' \tag{38}$$

or, setting $z - z' = \zeta$ and writing the impulse-response function in the form introduced in (31),

$$H(z;\ k) = \int_0^z h(z;\ \zeta)\ e^{-ik\zeta}d\zeta \tag{39}$$

The last relation shows that the frequency-response function of the vehicle may be defined at each height z as the Fourier transform in ζ of the impulse-response function $h(z;\ \zeta)$.

We shall now define the generalized power spectrum of the wind-velocity field as the double Fourier transform of the covariance function of the field.[6] We write

$$\Phi_{uu}(k_1,\ k_2) = \int_0^\infty\int_0^\infty \mathscr{C}_{uu}(z_1,\ z_2)\ e^{i(k_1 z_1 - k_2 z_2)}dz_1 dz_2 \tag{40}$$

The inverse transformation yields

$$\mathscr{C}_{uu}(z_1,\ z_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^\infty\int_{-\infty}^\infty \Phi_{uu}(k_1,\ k_2)\ e^{i(k_2 z_2 - k_1 z_1)}dk_1 dk_2 \tag{41}$$

Substituting for $\mathscr{C}_{uu}(z_1',\ z_2')$ from (41) into (24), and using (38), we express the covariance function of the response as a double convolution of the frequency-response function of the vehicle and the generalized power spectrum of the wind field

$$\mathcal{C}_{qq}(z_1, z_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H^*(z_1; k_1)\, H(z_2; k_2)\, \Phi_{uu}(k_1, k_2)\, e^{i(k_2 z_2 - k_1 z_1)}\, dk_1 dk_2 \qquad (42)$$

where $H^*(z; k)$ denotes the conjugate of $H(z; k)$.

Substituting for $\mathcal{C}_{qq}(z_1, z_2)$ from (42) into the relations (14), we obtain the following expressions for the functions $d_{ij}(z)$:

$$d_{11}(z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H^*(z; k_1)\, H(z; k_2)\, \Phi_{uu}(k_1, k_2)\, e^{i(k_2 - k_1)z}\, dk_1 dk_2 \qquad (43)$$

$$d_{12}(z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [H_z^*(z; k_1) - ik_1 H(z; k_1)]\, H(z; k_2)\, \Phi_{uu}(k_1, k_2)\, e^{i(k_2 - k_1)z}\, dk_1 dk_2 \qquad (44)$$

$$d_{22}(z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [H_z^*(z; k_1) - ik_1 H(z; k_1)]\, [H_z(z; k_2) + ik_2 H(z; k_2)]\, \Phi_{uu}(k_1, k_2)\, e^{i(k_2 - k_1)z}\, dk_1 dk_2 \qquad (45)$$

where $H_z(z; k)$ denotes the derivative of the frequency-response function with respect to z.

An alternate expression for the functions $d_{ij}(z)$, involving a simple convolution followed by a single Fourier transformation, may be obtained by introducing the new coordinates

$$z = \frac{z_1 + z_2}{2}, \qquad \eta = z_2 - z_1 \tag{46}$$

Noting that, after this change of variables, the exponent in Eq. (40) takes the form

$$i(k_1 z_1 - k_2 z_2) = -i[(k_2 - k_1)z + \frac{k_1 + k_2}{2}\eta]$$

we introduce the new wave-numbers

$$k = \frac{k_1 + k_2}{2}, \qquad \kappa = k_2 - k_1 \tag{47}$$

Solving Eqs. (46) and (47) for the old variables, substituting into Eq. (40), and introducing the new functions

$$\begin{aligned} \mathcal{C}_{uu}(z;\ \eta) &= \mathcal{C}_{uu}(z - \frac{\eta}{2},\ z + \frac{\eta}{2}) \\ \Phi_{uu}(k;\ \kappa) &= \Phi_{uu}(k - \frac{\kappa}{2},\ k + \frac{\kappa}{2}) \end{aligned} \tag{48}$$

we obtain the relation

$$\Phi_{uu}(k;\ \kappa) = \int_0^\infty \int_0^\infty \mathcal{C}_{uu}(z;\ \eta) e^{-i(\kappa z + k\eta)} d\eta dz \tag{49}$$

which provides an alternate way for computing the generalized power spectrum. We note the following property of symmetry:

$$\Phi_{uu}(k;\ -\kappa) = \Phi^*_{uu}(k;\ \kappa) \tag{50}$$

Carrying the new variables (46) and (47) into Eqs. (43), (44), and (45), we obtain the following expressions for the functions $d_{ij}(z)$:

$$d_{ij}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} D_{ij}(z;\ \kappa)\ e^{i\kappa z} d\kappa \tag{51}$$

where

$$D_{11}(z;\ \kappa) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H^*(z;\ k - \frac{\kappa}{2})\ H(z;\ k + \frac{\kappa}{2})\ \Phi_{uu}(k;\ \kappa) dk \tag{52}$$

$$D_{12}(z;\ \kappa) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [H^*_z(z;\ k - \frac{\kappa}{2}) - i(k - \frac{\kappa}{2})\ H^*(z;\ k - \frac{\kappa}{2})]$$

$$H(z;\ k + \frac{\kappa}{2})\ \Phi_{uu}(k;\kappa) dk \tag{53}$$

$$D_{22}(z;\ \kappa) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [H^*_z(z;\ k - \frac{\kappa}{2}) - i(k - \frac{\kappa}{2})\ H^*(z;\ k - \frac{\kappa}{2})]$$

$$[H_z(z;\ k + \frac{\kappa}{2}) + i(k + \frac{\kappa}{2})\ H(z;\ k + \frac{\kappa}{2})]\ \Phi_{uu}(k;\ \kappa) dk \tag{54}$$

The functions $d_{ij}(z)$ may thus be obtained by a single Fourier transformation from the functions $D_{ij}(z;\ \kappa)$, which are themselves obtained by forming simple convolutions of the generalized power spectrum and of the frequency-response function and its derivative. We note that, while the functions $D_{ij}(z;\ \kappa)$ are not the Fourier transforms of the functions

$d_{ij}(z)$ - since they are themselves dependent upon z - they satisfy the symmetry relation

$$D_{ij}(z; -\kappa) = D^{*}_{ij}(z; \kappa) \qquad (55)$$

which will help in reducing the length of the numerical computations.

APPENDIX

To obtain a better understanding of the error involved in the use of the approximation (4)

$$P_\alpha(z) = p_\alpha(z) \; [-\int_0^z p_\alpha(r)\, dr] \qquad \text{(A-1)}$$

for the probability density of a first occurence of the value $q = \alpha$, we shall compare this approximation with the exact value of $P_\alpha(z)$. This value may be expressed as a series of integrals through a straightforward application of the inclusion and exclusion method as indicated in [4]:

$$P_\alpha(z) = p_\alpha(z) - \frac{1}{1!} \int_0^z p_\alpha(r_1, z)\, dr_1$$

$$+ \frac{1}{2!} \int_0^z \int_0^z p_\alpha(r_1, r_2, z) dr_1 dr_2 + \ldots \qquad \text{(A-2)}$$

where $p(r_1, r_2, r_3, \ldots, r_n, z)\; dr_1 dr_2 \ldots\, dr\, dz$ is the probability of crossings of $q = \alpha$ with a positive slope in intervals r_1, $r_1 + dr_1$; r_2, $r_2 + dr_2$; ...; r_n, $r_n + dr_n$, and z, $z + dz$.

Expanding the exponential in (4) in a series, we obtain

$$p_\alpha(z)\ \exp\left[-\int_0^z p_\alpha(r)dr\right] = p_\alpha(z) - \frac{1}{1!}\int_0^z p_\alpha(z)p_\alpha(r_1)dr_1$$

$$+ \frac{1}{2!}\int_0^z\int_0^z p_\alpha(z)\ p_\alpha(r_1)\ p_\alpha(r_2)dr_1dr_2 - \ldots \qquad \text{(A-3)}$$

Subtracting (A-3) from (A-2) term by term yields the difference between the exact and approximate expressions for $P_\alpha(z)$

$$P_\alpha(z) - p_\alpha(z)\ \exp\left[-\int_0^z p_x(r)dr\right] =$$

$$- \frac{1}{1!}\int_0^z \left[p_\alpha(r_1,z) - p_\alpha(z)\ p_\alpha(r_1)\right] dr_1$$

$$+ \frac{1}{2!}\int_0^z\int_0^z\left[p_\alpha(r_1, r_2, z) - p_\alpha(z)p_\alpha(r_1)p_\alpha(r_2)\right]dr_1dr_2 - \ldots$$

$$\text{(A-4)}$$

We thus check that the error in the approximation (4) is due to neglecting the dependence of an α-crossing at a particular height on previous α-crossings since, if the crossings are independent,

$$p_\alpha(r_1, r_2, \ldots, r_n, z) = p_\alpha(r_1)\ p_\alpha(r_2)\ \ldots\ p_\alpha(r_n)\ p_\alpha(z).$$

This appears to be a reasonable assumption for our particular application since one would expect, for large values of α, that the α-crossings would indeed be rare events and as such could be treated as independent events.

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